OPTIMA 86

Mathematical Optimization Society Newsletter

Philippe L. Toint

MOS Chair's Column

July 1, 2011. For the few sorry spirits who, misguided by the general state of the world, thought that mathematical optimization was in theoretical decline or too remote from applications in the real world, the SIOPT Conference on Optimization in Darmstadt in May was a real and vivid counterexample. Indeed this very well attended meeting (600+ participants) was a resounding success in terms of interest and quality of the talks. As has been so far the case in this series of meetings, the focus was mainly on continuous problems: in particular, problems arising from continuous mechanics, fluids and control were prominent, showing the very healthy state of not only optimization in those domains, but also the German industry's interest in optimization in general. The significant presence of discrete optimization was also noticeable, with several interesting sessions and plenary talks in this area. As optimizers, all were delighted that, once more, a high quality conference has been organized by SIAM in addition to the major events organized by the Mathematical Optimization Society.

If mathematical optimization is turning today into one of the major branches in applied mathematics, this is due not only to our present efforts as scientists, but also to those of the founding fathers of our research domain. One of them, Charles Broyden (the B in BFGS) unfortunately passed away on Friday 20th May, at the age of 78 (see the obituary published in this issue on page 10). His memory will stay with us for long, and his work will undoubtedly continue to inspire.

The beginning of 2011 was also the time to start thinking about the various prizes sponsored by MOS, which will be awarded in the *International Mathematical Programming Symposium* in Berlin in August 2012. It may be useful to recall that the MOS currently awards five scientific prizes and a named lectureship. These are the Dantzig, Lagrange, Beal-Orchard-Hays, Fulkerson and Tucker prizes, and the Paul Tseng Lectureship, whose more complete description, scope and past winners can be found on the MOS Website (http://www.mathprog.org). The respective committees have now been established for deciding to whom these distinctions must be given, and I would like to take this opportunity to thank all of our colleagues who kindly accepted to serve on these committees. I would also like to call on all members to think about proposing high quality submissions for these prizes. I am certain that their scientific value can only be enhanced by friendly competition between high quality submissions. I am personally looking forward to meeting you all at the the award ceremony during the opening session of the Berlin ISMP.

This is also the time to start looking at possible sites that will host ISMP in 2015. The call for proposal submission can be found in this issue on page 12. And as always, do not forget to renew your MOS membership.

Meanwhile, enjoy the summer (for the majority of us in the northern hemisphere) and let us keep the abundance and quality of our scientific activities at the present vibrant level.

Note from the Editors

The stable set problem in claw-free graphs is the main topic of this issue of Optima. Much of the tremendous progress that has recently been obtained on this generalization of the matching problem is due to work of the authors Gianpaolo Oriolo, Gautier Stauffer, and Paolo Ventura of the article you'll find below and their co-workers. In the discussion column, Manfred Padberg shares with us his memories of the historical context in which the interest in the stable set problem in claw-free graphs arose and of how it traveled to Italy.

Katya Scheinberg, Editor
Sam Burer, Co-Editor
Volker Kaibel, Co-Editor

Contents of Issue 86 / July 2011

1 Philippe L. Toint, *MOS Chair's Column*
1 Note from the Editors
1 Gianpaolo Oriolo, Gautier Stauffer and Paolo Ventura, *Stable Sets in Claw-Free Graphs: Recent Achievements and Future Challenges*
8 Manfred W. Padberg, *Node Packings in Graphs and Claw Free Graphs*
10 Oleg Burdakov, John Dennis, and Jorge Moré, *Charles G. Broyden, 1933–2011*
10 Call for Nominations for the 2012 Beale-Orchard-Hays Prize
11 ISMP 2012 in Berlin
12 Imprint

Gianpaolo Oriolo, Gautier Stauffer and Paolo Ventura

Stable Sets in Claw-Free Graphs: Recent Achievements and Future Challenges

1 Introduction

A stable set in a graph $G(V,E)$ is a set of vertices that are pairwise non-adjacent. When G is the intersection graph of the edges of a graph H – two edges intersect if they share an endpoint – a stable set in G corresponds to a matching in H (and vice-versa). Hence while the stable set problem is hard in general, the special case of line graphs – the family of all such intersection graphs – can be handled in polynomial time through matching.

Matching is a classic problem in combinatorial optimization and it exhibits some remarkable properties. Many of those properties have been extended and led to very powerful tools and theories like for instance matroid intersection or delta-matroids. In order to extend the matching theory to the stable set setting, it appears that two fundamental properties of matching are crucial: *the augmenting path property* and the *intersection property*.

Petersen observed in 1891 (and Berge proved in 1952) that the symmetric difference of two matchings is made of alternating paths and even cycles. In particular, a matching M is of maximum cardinality in a graph G if and only if there does not exist any augmenting path in G with respect to M. Moreover, as two matchings are adjacent on the matching polytope $MATCH(G)$ – the convex hull of all incidence vectors of matchings in a graph G – if and only if they have a connected symmetric difference, one can easily show that $MATCH(G)$ has the intersection property: $MATCH(G) \cap \{x : \sum_{e \in E} x_e = k\}$ is integral for every integer k.

Interestingly those properties extend to the stable set setting beyond line graphs (alternating paths and cycles being defined in terms of vertices here): they are also valid for stable sets in *claw-free graphs* – a graph is claw-free if no vertex has a stable set of size three in its neighborhood. This was observed by Berge in 1973 for the symmetric difference of stable sets in claw-free graphs and by Calvillo in 1979 for the intersection property. Remarkably, Berge and Calvillo also proved the converse, i.e., a class of graphs exhibits one or the other of those properties for the stable set problem if and only if it is a subclass of claw-free graphs. Hence, with respect to stable sets, claw-free graphs appear to be the right framework to extend the aforementioned properties of matching. The problem of finding a maximum weighted stable set in claw-free graphs has been therefore investigated by several people, and its theory has been developing for more than 40 years. The last 10 years have been particularly productive, mainly due to new approaches that exploit results from structural graph theory. The purpose of this paper is to help the interested researchers to navigate through the various results in this field and in particular to shed light on the latest achievements and the current open questions.

For the sake of shortness, some theorems might be slightly imprecise. In this case, a reference is given, and the reader should rely on that. Also we often denote by $V(G)$ and $E(G)$ the vertex set and the edge set of a graph G.

2 Stable Sets in Claw-Free Graphs: Some Classical Results

In this section we survey a few classical results on the problem. We first deal with some algorithmic results, and then move to some polyhedral questions.

2.1 Algorithms for the Maximum Weighted Stable Set Problem

Given a claw-free graph $G(V,E)$ and a weight function $w : V \mapsto \mathbb{R}$, a maximum weighted stable set (MWSS) can be found in polynomial time. We denote by $\alpha(G)$ the cardinality of such a stable set when w is the all ones vector; $\alpha(G)$ is also called the *stability number* of G. At the present time, there are several algorithms for the problem, and we may recognize three different main approaches. A first class of algorithms deals with augmenting paths techniques, and the algorithms by Minty [35] and Sbihi [50], respectively, for the weighted and the unweighted case, follow this approach. In fact, as we already discussed, Berge's augmenting path theorem for matching extends to stable sets in claw-free graphs (a path P is *augmenting* with respect to a stable set S if $(V(P) \setminus S) \cup (S \setminus V(P))$ is a stable set of size $|S|+1$):

Theorem 1 ([4]). *A stable set S is maximum for a claw-free graph G if and only if there are no paths that are augmenting with respect to S.*

Sbihi's algorithm builds upon this theorem while Minty's builds upon a cute extension to the weighted case (given an augmenting path P with respect to a stable set S, the weight of this path is given by $w(V(P) \setminus S) - w(V(P) \cap S)$):

Theorem 2 ([35]). *Let S be a MWSS of size k, and let P be an augmenting path of maximum weight with respect to S. Then $(S \setminus V(P)) \cup (V(P) \setminus S)$ is a MWSS of size $k+1$.*

Minty's idea is to detect those maximum weight augmenting paths and proceed with at most $|V|$ augmentations. Given two "exposed" vertices u, v of $V \setminus S$, i.e., they are both adjacent to a single vertex of S, Minty's crucial idea is that of reducing the problem of finding an $u-v$ augmenting path with maximum weight to the problem of finding a matching with maximum weight in an auxiliary graph H. The construction of H is rather intricate. We simply mention here that this graph has $O(|V|)$ vertices. Hence the whole algorithm requires the solution of $O(|V|^3)$ weighted matching problems in an auxiliary graph with $O(|V|)$ vertices.

In 2001 the algorithm of Minty was slightly revised by Nakamura and Tamura [36], as they realized that, in the weighted case, the algorithm could fail for some special configurations. Subsequently, Schrijver [51], elaborating on Minty's algorithm, proposed an elegant alternative using a slightly different edge-weighted auxiliary graph H. The algorithm can be implemented to run in time $O(|V|^5 \log|V| + |V|^4|E|)$ in the weighted case and in time $O(|V|^5)$ in the unweighted one (however, Sbihi claimed that her algorithm, for the unweighted case, can be implemented to run in time $O(|V|^3)$.

An entirely different approach, based on reduction techniques, was taken by Lovász and Plummer [34], for solving the problem in the unweighted case. The crucial idea here is that of performing a series of graph reductions that preserve the stability number, as to end up with a line graph, where one has to solve a *single* matching problem. The resulting algorithm is very elegant, much less intricate than the previous algorithms, and, as Lovász and Plummer point out, with some care it can be implemented as to run in $O(|V|^4)$. Unfortunately, in spite of some efforts, it is not clear how to extend this algorithm to the weighted case.

However, recently Nobili and Sassano [39] were able to combine ideas from both the algorithm of Minty and that of Lovász and Plummer to provide a new algorithm for the weighted case that runs in $O(|V|^4 \log|V|)$-time. If we compare (very roughly!) their algorithm with Minty's algorithm, we see that, on one hand Nobili and Sassano are able to reduce the number of matching problems that have to be solved to $O(|V|^2)$, while on the other they are able to solve each of these problem in $O(|V|^2 \log|V|)$-time, thanks to a weighted reduction, inspired from that of Lovász and Plummer.

A latter solution approach to the MWSS problem in claw-free graphs is based on decomposition techniques and has been taken by Oriolo, Pietropaoli and Stauffer [41] first, and by Faenza, Oriolo, and Stauffer [19] later. The latter algorithm can be implemented to run in time $\mathcal{O}(|V|(|V|\log|V| + |E|))$. We postpone the discussion about these algorithms to Section 4.1, as it is first convenient to deal with some structural decomposition results for claw-free graphs.

2.2 Stable Sets in Claw-Free Graphs: Polyhedral Issues

The stable set polytope $STAB(G)$ of a graph $G(V,E)$ is the convex hull of the characteristic vectors of stable sets in G, i.e., $STAB(G) = conv\{x \in \{0,1\}^{|V|} \mid x_u + x_v \leq 1, \forall \{u,v\} \in E\}$. Since the seminal paper by Padberg [43], this polytope has been carefully investigated by several authors (see e.g. [12, 37, 38]).

Because the MWSS problem in claw-free graphs can be solved in polynomial time, exact separation over this polytope also can be done in polynomial time [28], and hence the stable set polytope of claw-free graphs is somewhat "under control". However no complete linear description is known at the time of writing, despite the fact that the problem is "officially" open for more than a quarter of a century [29]: "in spite of considerable efforts, no decent system of inequalities describing "STAB(G)" for claw-free graphs is known". Such a description would possibly result in a nice minmax characterization of the problem.

A neat description is at hand for the stable set polytope of line graphs. Indeed, Edmonds [15] proved that the matching polytope $MATCH(H)$ of a graph $H(V,E)$ – i.e., $conv\{x \in \{0,1\}^{|E|} | \sum_{e \in \delta(v)} x_e \leq 1, \forall v \in V\}$ – can be described by non-negativity inequalities, degree inequalities (as usual, we denote by $\delta(v)$ the set of edges incident to a node v), and odd sets inequalities, where, for an odd set $S \subseteq V$, we denote by $E(S)$ the set of edges between vertices of S.

Theorem 3 ([15]). *The matching polytope of a graph $H(V,E)$ can be characterized as $MATCH(H) = \{x \in \mathbb{R}^{|E|} | x \geq 0; \sum_{e \in \delta(v)} x_e \leq 1, \forall v \in V; \sum_{e \in E(S)} x_e \leq \lfloor \frac{|S|}{2} \rfloor\}$, for every odd set $S \subseteq V\}$.*

But since $MATCH(H) = STAB(L(H))$, where $L(H)$ denotes the line graph of H, it follows that the stable set polytope of line graphs can be described by non-negativity inequalities, clique inequalities and *Edmonds' inequalities*, the counterpart of odd set inequalities in the stable set setting. More formally:

Definition 2.1. *For a graph $G(V,E)$ and an odd set of cliques $\mathcal{K}$, let $V_{\geq 2}(\mathcal{K})$ be the set of vertices covered by at least 2 cliques of $\mathcal{K}$. The* Edmonds' inequality *associated with $\mathcal{K}$ is: $\sum_{v \in V_{\geq 2}(\mathcal{K})} x_v \leq \lfloor \frac{|\mathcal{K}|}{2} \rfloor$.*

From Definition 2.1 it follows that Edmonds' inequalities are derived as Chvátal-Gomory cuts from the *clique relaxation* of the stable set polytope $QSTAB(G) := \{x \in \mathbb{R}^{|V|} : x \geq 0; x(K) \leq 1$, for every clique K of $G\}$ [17], i.e., they can be obtained by first taking a non-negative combination of the inequalities describing $QSTAB(G)$, and then rounding down the right hand side of the combination.

Lemma 2.1 ([15]). *For a line graph G, non-negativity inequalities, clique inequalities and Edmonds' inequalities are enough to describe the stable set polytope.*

Unfortunately, Lemma 2.1 does not hold true for claw-free graphs. In fact, consider a 5-wheel, i.e., a graph with vertex set $\{w, v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{(w, v_i), (v_i, v_{i+1})$ for all $i = 1, .., 5\}$ with $v_6 \equiv v_1$, then the *5-wheel inequality* $\sum_{i=1}^{5} x_{v_i} + 2x_w \leq 2$ is a facet of its stable set polytope. This shows that *non-rank* inequalities are needed in order to define the stable set polytope of claw-free graphs. An inequality is *rank* if it only involves {0,1}-valued coefficients in the left hand side, i.e., if it is of the form $\sum_{v \in S} x_v \leq \alpha(G[S])$ for $S \subseteq V$.

In 1978 Maurras, inferring that 5-wheels and, more generally, odd-antiwheels (i.e., a graph made of a vertex totally joined to the complement of an odd hole) were the problem, introduced the class of *quasi-line graphs*, i.e., claw-free graphs without odd-antiwheels. He also conjectured that for quasi-line graphs all facets of $STAB(G)$ are rank. Building upon Maurras' conjecture, Sbihi conjectured that for claw-free graphs all facets of $STAB(G)$ have only {0,1,2}-valued coefficients. Both conjectures were proven false by Giles and Trotter [27] in 1981 (see Figure 2 and 1, respectively). We know now that for claw-free graphs with stability number 3 there exist facets with arbitrarily many coefficients [46] and that for any integer a there exist quasi-line graphs whose stable set polytopes involve facets with coefficient a and $a + 1$ [27, 33]. While Maurras' conjecture was wrong, his intuition on the relevance of the class of quasi-line graphs was correct. Indeed, in contrast with general claw-free graphs,

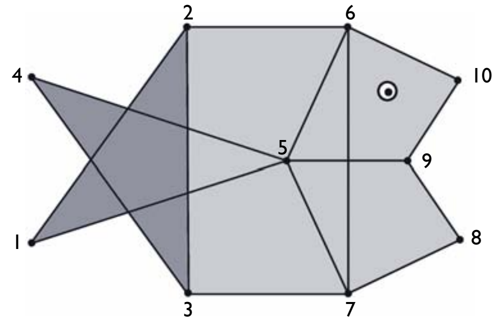

Figure 1. *The complement of a claw-free graph G. The graph G induces the facet: $2x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 + x_6 + x_7 + 3x_8 + 3x_9 + 3x_{10} \leq 4$. Note that G is not quasi-line and that $\alpha(G) = 3$. (The picture is a courtesy of Tristram Bogart, Annie Raymond and Rekha Thomas.)*

the nature of the inequalities needed for quasi-line graphs was grasped first by Ben Rebea [49] and later by Oriolo [40] who named a conjecture after him: the Ben Rebea conjecture.

The Ben Rebea Conjecture 1 (Oriolo [40]). *For a quasi-line graph G, non-negativity inequalities, clique inequalities and clique family inequalities are enough to describe $STAB(G)$.*

Definition 2.2. *Given a graph G, a family of cliques $\mathcal{K}$ and an integer $p \geq 2$, define $V_{\geq p}(\mathcal{K})$ and $V_{p-1}(\mathcal{K})$ as the set of vertices covered by at least p cliques and exactly $p - 1$ cliques, respectively. The following inequality is valid for $STAB(G)$ and is called the* clique family inequality *associated with $\mathcal{K}$ and p: $\sum_{v \in V_{\geq p}(\mathcal{K})} x_v + \frac{p-r-1}{p-r} \cdot \sum_{v \in V_{p-1}(\mathcal{K})} x_v \leq \lfloor \frac{|\mathcal{K}|}{p} \rfloor$, where $r = |\mathcal{K}| \mod p$.*

Clique family inequalities generalize Edmonds' inequalities, and their validity can easily be derived by the disjunction $\sum_{v \in V_{\geq p}(\mathcal{K}) \cup V_{p-1}(\mathcal{K})} x_v \leq \lfloor \frac{|\mathcal{K}|}{p} \rfloor \vee \sum_{v \in V_{\geq p}(\mathcal{K}) \cup V_{p-1}(\mathcal{K})} x_v \geq \lfloor \frac{|\mathcal{K}|}{p} \rfloor + 1$ applied to $QSTAB(G)$.

The Ben Rebea conjecture suggested that the stable set polytope of quasi-line graphs has a neat description. As for claw-free graphs, in 1991 Galluccio and Sassano [26] provided an elegant characterization of *rank minimal facets*, i.e., rank facets that are minimal with respect to lifting and complete join operations [11, 43].

We close this section by illustrating the result of Calvillo [5] that we mentioned before. Calvillo proved the following nice property of the stable set polytope of claw-free graphs. A polytope $P \subseteq \mathbb{R}^n$ has the *intersection property* if $P \cap \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = k\}$ is integral for all integer k.

Theorem 4 ([5]). *$STAB(G)$ has the intersection property if and only if G is a claw-free graph.*

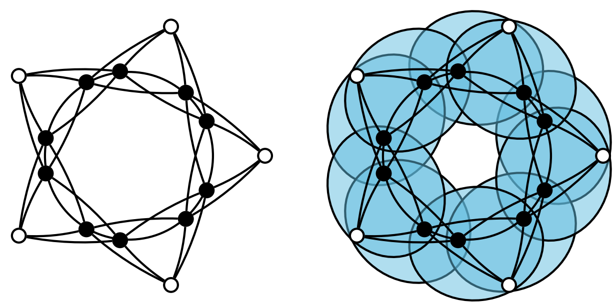

Figure 2. *A quasi-line graph inducing the facet $\sum_{v \in \circ} x_v + 2 \cdot \sum_{v \in \bullet} x_v \leq 6$. On the right, the cliques involved in the derivation of the inequality as a clique family inequality.*

3 A Breakthrough: Decomposition of Claw-Free and Quasi-Line Graphs

In a long series of paper, Chudnovsky and Seymour (see e.g. [8, 9, 10, 6]) elucidate the structure of claw-free graphs and define a decomposition result for them. For this purpose, they have introduced a new composition operation.

3.1 The Composition of Strips

In order to better grasp this operation, it is convenient to first deal with an algorithmic procedure that can be used to build line graphs. The rationale of this latter operation is the following. Given a graph G, each vertex in G can be associated with a clique in the line graph $H = L(G)$ (all edges incident to this vertex are pairwise adjacent in H). If we let $\mathcal{F}$ denote the family of cliques of H that are associated with vertices of G, we observe that $\mathcal{F}$ has the following properties: (i) every edge of H is covered by some clique of $\mathcal{F}$; (ii) every vertex of H is covered by exactly two cliques of $\mathcal{F}$.

Suppose now that we are given a (general) graph H. We call a family $\mathcal{F}$ of cliques of H a *Krausz family* if it satisfies the above properties. Krausz [32] proved the following:

Theorem 5 ([32]). *A graph is the line graph of a multi-graph if and only if it admits a Krausz family.*

This theorem gives an algorithmic procedure to build line graphs. This procedure requires as input a set of vertices V and a partition $\mathcal{P} = P_1, ..., P_q$ of the multi-set $V \cup V$. It then associates to the pair $(V, \mathcal{P})$ the graph G with vertex set V and edge set $E := \{\{u, v\} : u \neq v$ and both $u, v \in P_i$, for some $1 \leq i \leq q\}$. Chudnovsky and Seymour generalized the above construction, essentially by replacing *vertices* with *strips*. We borrow (but slightly change) some definitions of theirs.

Definition 3.1. *A* strip $(G, \mathcal{A})$ *is a graph* G *(not necessarily connected) with a multi-family* $\mathcal{A}$ *of either one or two designated non-empty cliques (possibly identical) of* G*. The cliques in* $\mathcal{A}$ *are called the* extremities *of the strip.*

Let $\mathcal{H} = \{(G^i, \mathcal{A}^i),\ i = 1, ..., k\}$ be a family of vertex disjoint strips. Let $\mathcal{A}(\mathcal{H})$ denote the multifamily of the extremities of those strips, i.e., $\mathcal{A}(\mathcal{H}) = \bigcup_{i=1..k} \mathcal{A}^i$, and let $\mathcal{P} = P_1, P_2, ..., P_q$ be the classes of a partition of $\mathcal{A}(\mathcal{H})$. We associate to the pair $(\mathcal{H}, \mathcal{P})$ the graph G that is made of the disjoint union of the graphs $G^1, \ldots G^k$, with additional edges $E := \{\{u, v\} : u \neq v$ and u and v belong to different extremities in a same class P_i, for some $1 \leq i \leq q\}$. G is called the composition of the strips $\mathcal{H}$ with respect to partition $\mathcal{P}$. Note that, for line graphs, this composition reduces to the above construction, as soon as each graph G^i is made of a single vertex v_i and the corresponding strip is $(\{v_i\}, \{\{v_i\}, \{v_i\}\})$.

Even though the operation of composition of strips builds graphs that are in general non-line, such graphs indeed inherit a "line structure" from its similarity with Krausz composition. Say that a strip $H = (G, \mathcal{A})$ is *line* if G admits a Krausz family $\mathcal{K}$ with $\mathcal{A} \subseteq \mathcal{K}$. Then, as soon as all *strips* are *line*, the composition is a line graph. The proof of this fact is straightforward. We will make heavy use of this fact in the following.

Lemma 3.1. *Let* G *be the composition of a family of* line *strips* $H^i = (G^i, \mathcal{A}^i), i = 1, ..., k$ *with respect to a partition* $\mathcal{P}$*. Then* G *is a line graph.*

3.2 Decomposition Results for Claw-Free and Quasi-Line Graphs

In [8] Chudnovsky and Seymour overview a series of papers in which they prove a structure theory for claw-free graphs. The theory is too complex to describe in detail here, so we just outline two of their results.

Theorem 6 ([10]). *Let* $G(V, E)$ *be a connected claw-free graph. Then one of the followings holds: i)* $\alpha(G) \leq 3$ *and* G *belongs to a small set of basic graphs; ii)* G *is a fuzzy circular interval graph; iii)* G *is the composition of strips, that are either fuzzy linear interval strips or they belong to one of a small number of family of strips, all with stability number at most* 3.

Circular interval graphs are defined by a set of vertices, a circle and a set of arcs. Vertices are mapped to the circle and two vertices are adjacent if and only if they are covered by an arc. Those graphs are also known as proper circular arc graphs. Linear interval graphs are constructed in the same way as circular interval graphs, but on a line rather than on a circle. In a linear interval strip $(G, \mathcal{A})$, G is a linear interval graph and the cliques in $\mathcal{A}$ are made of contiguous vertices at the end of the line segment. *Fuzzy circular/linear interval graphs* are a slight generalization of circular/linear interval graphs.

The results considerably simplify for the subclass of quasi-line graphs.

Theorem 7 ([8]). *Let* $G(V, E)$ *be a connected quasi-line graph. One of the followings holds:* G *is a fuzzy circular interval graph;* G *is the composition of fuzzy linear interval strips.*

We point out that while the two above results are not algorithmic, lighter versions of those have been recently algorithmized by Hermelin, Mnich, van Leeuwen and Woeginger [30].

A different algorithmic decomposition theorem for claw-free graphs was recently given by Faenza, Oriolo and Stauffer. From a structural point of view, this result is much weaker than Theorem 6; however, it is particularly useful when dealing with the MWSS problem, as we discuss in Section 4.1.

Theorem 8 ([19]). *Let* $G(V, E)$ *be a claw-free graph. In time* $\mathcal{O}(|V||E|)$*, one can find out whether* $\alpha(G) \leq 3$*, or* G *is almost nearly distance simplicial, or* G *is the composition of* $\mathcal{O}(|V|)$ *strips that are distance simplicial strips or strips with stability number at most 3 and containing a* 5*-wheel (and provide the decomposition).*

We denote by $N(S)$, with $S \subset V$, the set of nodes of $V \setminus S$ that are adjacent to some node in S (we also use the notation $N(v)$ for $N(\{v\})$). A connected graph G is *distance simplicial with respect to a clique* K if, for every j, $\alpha(N_j(K)) \leq 1$, i.e., each neighborhood $N_j(K)$ of K is a clique; if there exists a vertex v such that G is distance simplicial with respect to $\{v\}$, we simply say that G is *distance simplicial*. A *distance simplicial strip* is a strip $(G, \mathcal{A})$, such that G is distance simplicial with respect to each clique in $\mathcal{A}$. A graph is *nearly* distance simplicial if, for each $v \in V$, $G \setminus (N(v) \cup \{v\})$ is distance simplicial. *Almost* nearly distance simplicial graphs are a slight generalization. When G is quasi-line, Theorem 8 reads as follows:

Theorem 9 ([19]). *Let* $G(V, E)$ *be a quasi-line graph. In time* $\mathcal{O}(|V||E|)$*, one can find out whether* G *is the composition of* $\mathcal{O}(|V|)$ *distance simplicial strips, or* G *is almost nearly distance simplicial (and provide the decomposition).*

4 Following the Breakthrough: Stable Sets in Claw-Free Graphs Revisited

4.1 Faster Algorithm for Claw-Free Graphs

Suppose that we are interested in solving the MWSS problem on a graph G that is the composition of some strips $H^1, \ldots, H^k$ and that, in particular, we are able to solve the same problem on each strip. Building upon Lemma 3.1 we may reduce the former problem to a matching problem. This goes as follows: we replace each strip $H^i, i = 1, ..., k$ with suitable, simple, line strips $\underline{H}^i, i = 1, ..., k$, and consider the graph $\underline{G}$ obtained by substituting H^i with $\underline{H}^i$ in the composition. Following Lemma 3.1, $\underline{G}$ is a line graph, and therefore a MWSS of $\underline{G}$ can be found by solving a matching problem. Finally, from a MWSS of $\underline{G}$ we then recover a MWSS of G. We have in fact:

Theorem 10 ([41]). *The maximum weighted stable set problem on a graph G that is the composition of some strips $(G^1, \mathcal{A}^1), \ldots, (G^k, \mathcal{A}^k)$ can be solved in $O(|V(G)|^2 \log |V(G)| + \sum_{i=1,\ldots,k} p_i(|V(G^i)|))$-time, if each G^i belongs to some class of graphs, where the same problem can be solved in time $O(p_i(|V(G^i)|))$.*

Faenza, Oriolo and Stauffer [19] recently proposed a strongly polynomial algorithm for solving the MWSS problem in a claw-free graph $G(V, E)$ that runs in $\mathcal{O}(|V|(|E| + |V| \log |V|))$−time, drastically improving the previous best known complexity bound. This algorithm builds upon Theorem 8, and, in the following, we sketch how it deals with the different cases arising from that theorem. Let G be a claw-free graph. If $\alpha(G) \leq 3$, then a MWSS can be found by enumeration. If G is the composition of strips, then the result follows from Theorem 10, as soon as we observe that we can find a MWSS in a distance simplicial strip by dynamic programming, following a construction and an algorithm from Pulleyblank and Shepherd [48] for distance-claw-free graphs. The latter construction can be used also when G is almost nearly distance simplicial. Without using any sophisticated data structures, the algorithm can be implemented as to run in $\mathcal{O}(|V|(|E| + |V| \log |V|))$−time.

4.2 The Stable Set Polytope of Quasi-Line Graphs

When studying the polyhedral aspect of a composition of graphs, it is standard to substitute some of the graphs with "gadgets" and to derive the polyhedral description of the composition from the polyhedral descriptions of the composition of the simpler graphs [2, 3]. For the composition of strips, Chudnovsky and Seymour observed [7, 53] that paths of length one or two were the appropriate gadgets to prove the following:

Theorem 11 ([7, 53]). *The stable set polytope of a quasi-line graph G that is not a fuzzy circular interval graph can be characterized by non-negativity inequalities, clique inequalities and Edmonds' inequalities.*

On one hand this theorem shows that the Ben Rebea conjecture holds for such class of quasi-line graphs (Edmonds' inequalities are particular clique family inequalities). On the other, with the help of Theorem 7, it shows that all non-rank facet inducing inequalities for quasi-line graphs appear in fuzzy circular interval graphs.

Eisenbrand, Oriolo, Stauffer and Ventura [17] were able to provide a linear description of $STAB(G)$, when G is a fuzzy circular interval graph. As fuzziness can be handled easily, in the following we simply deal with circular interval graphs. For such graphs, let A be the clique incidence matrix, when one restricts to cliques stemming from the intervals. Then the stable set problem can be formulated as: $\max \{\sum_{v \in V} c_v x_v \; : \; Ax \leq 1 \; ; \; x_v \in \{0, 1\}, \forall v \in V\}$. What is crucial is that the matrix A has the so-called circular one property, i.e., there is an ordering of the columns such that, on each row, the ones appear consecutively, under the convention that the first column is consecutive to the last. But then the linear relaxation $P = \{x \in \mathbb{R}^n \mid \binom{A}{-I} x \leq \binom{1}{0}\}$ is such that $P^k = P \cap \{x : \sum_{v \in V} x_v = k\}$ is integral for any integer k (using the equation $\sum_{v \in V} x_v = k$, the system $\binom{A}{-I} x \leq \binom{1}{0}$ can be rewritten as a consecutive one system and so P^k is described by a totally unimodular system). This result shows that the only missing inequalities are disjunctive cuts of the form $\sum_{v \in V} x_v \leq k \vee \sum_{v \in V} x_v \geq k + 1$ from $QSTAB(G)$. Careful analysis of those disjunctive cuts allows one to prove that they are clique family inequalities and therefore the Ben Rebea conjecture holds true [17].

Theorem 12 ([17]). *Conjecture 1 holds true.*

4.3 The Stable Set Polytope of Claw-Free Graphs with $\alpha \geq 4$ and No Clique Cutsets

Galluccio, Gentile and Ventura [22] extended Theorem 11 to deal with the stable set polytope of a graph that is the composition of *arbitrary* strips. Let G be the composition of a family of strips $H^i = (G^i, \mathcal{A}^i), i = 1, .., k$. Strips with only one extremity can be easily handled because of a result of Chvátal [12]. Therefore assume without loss of generality that each strip has two extremities. We denote by G^i_z the graph obtained from G^i by adding a new node z with $N(z) = A^i_1 \cup A^i_2$, and by G^i_{uv} the graph obtained from G^i by adding two new nodes u and v such that $N(u) = A^i_1 \cup \{v\}$ and $N(v) = A^i_2 \cup \{u\}$. In [22] it is proved that the inequalities needed to describe $STAB(G)$ can be obtained by (appropriately) replacing the inequalities defining $STAB(G^i_{uv})$ and $STAB(G^i_z)$ in the stable set polytope of a certain line graph $\bar{G}$, derived from G by substituting each H^i with a line strip.

In the following, we apply this result to claw-free graphs. Galluccio, Gentile and Ventura [24, 25] managed to provide a description of the stable set polytope of the graphs G^i_z and G^i_{uv} associated with the strips $(G^i, \mathcal{A}^i)$ arising from Theorem 6: in particular, they showed that non-negativity inequalities, rank inequalities, sequential liftings [43] of 5-wheel inequalities and sequential liftings of geared inequalities are sufficient to describe both $STAB(G^i_z)$ and $STAB(G^i_{uv})$. We have therefore:

Theorem 13 ([25]). *The stable set polytope of a claw-free graph with stability number at least 4, non-fuzzy circular interval and with no clique-cutset is defined by: non-negative inequalities, sequential liftings of multiple geared inequalities, rank inequalities and sequential liftings of 5-wheel inequalities. In particular, all inequalities are $\{0, 1, 2\}$-valued.*

We recall that a description of rank inequalities in claw-free graphs follows from the characterization of *rank minimal facets* in [26]. As for *(multiple) geared inequalities* [21, 23], in claw-free graphs they are $\{0, 1, 2\}$-valued facet defining inequalities that are "produced" from rank inequalities, by substituting one or multiple edges with a *gear*, a graph that is made of two intertwined 5-wheels (see Figure 3).

Combining Theorem 6, Theorem 12 and Theorem 13, we have:

Theorem 14 ([25]). *The stable set polytope of any claw-free graph G without a clique-cutset and such that $\alpha(G) \geq 4$ is defined by: non-negativity inequalities, clique-family inequalities, rank inequalities, sequential liftings of 5-wheel inequalities, and sequential liftings of multiple geared inequalities.*

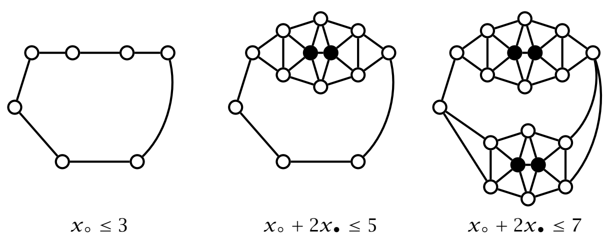

Figure 3. *A rank, a geared and a multiple geared facet defining inequality*

4.4 Extended Formulation and Separation for the Stable Set Polytope of Claw-Free Graphs

Faenza, Oriolo and Stauffer [19] gave a characterization of $STAB(G)$, G claw-free, in an extended space. Their result builds upon a suitable extended description of $STAB(G)$ for a graph G

that is the composition of a family of strips $H^i = (G^i, \mathcal{A}^i), i = 1, .., k$. In fact, Theorem 10 has a polyhedral interpretation in an extended space. We proceed as in Section 4.1, and let $\underline{G}$ be the line graph that arises by substituting each strip H^i with a suitable line strip $\underline{H}^i$ in the composition. Given a polytope $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ in $\mathbb{R}^n$, we call $\{(x, \lambda_P) \in \mathbb{R}^n \times \mathbb{R} : Ax \leq \lambda_P b, \lambda_P \geq 0\}$ the homogenized cone associated with P and the system $Ax \leq \lambda_P b, \lambda_P \geq 0$ the homogenization of the system $Ax \leq b$.

Theorem 15 ([18]). *An extended formulation for $STAB(G)$ is (mainly) given by the homogenization of the (possibly extended) linear descriptions of $STAB(G^i)$, $STAB(G^i[V(G) \setminus \cup_{A \in \mathcal{A}^i} A])$ and $STAB(G^i[V(G) \setminus A])$ for all $A \in \mathcal{A}^i$; and $STAB(\underline{G})$ where $\underline{G}$ is a line graph with $\mathcal{O}(k)$ vertices.*

(The reader should rely on [18] to find out what is behind the word "mainly" in the previous statement.) We now sketch how to apply this construction to a claw-free graph $G(V, E)$. By Theorem 8, we know that in time $O(|V||E|)$ we can distinguish if $\alpha(G) \leq 3$, if G is almost nearly distance simplicial or if G is the composition of distance simplicial strips and strips with stability number at most 3.

It is easy to write an extended formulation for a graph G with small stability number. Indeed, let $x_1, \ldots, x_k$ be all the extreme points of $STAB(G)$ i.e., all stable set of size $0, 1, 2$ or 3. The polytope $Q = \{(x, \lambda) : x = \sum_{i=1}^{k} \lambda_i x_i, \lambda \geq 0, \sum_{i=1}^{k} \lambda_i = 1\}$ is an extended formulation of $STAB(G)$. Nearly distance simplicial graphs are distance claw-free graphs, a class of graphs for which Pulleyblank and Shepherd [48] gave a compact extended formulation based on a dynamic programming algorithm for the MWSS problem (note that this class also includes graphs that are distance simplicial graphs with respect to some clique). We are left with the case where G is the composition of a family of strips $H^i = (G^i, \mathcal{A}^i), i = 1, .., k$. In this case, building upon Theorem 15, we just need to show that we are able to derive extended formulations for the stable set polytopes associated with the strips. In fact, if either G^i is a distance simplicial graph with respect to some clique, or $\alpha(G^i) \leq 3$, then an extended formulation for $STAB(G^i)$ (or $STAB(G^i[V(G) \setminus \cup_{A \in \mathcal{A}^i} A])$ etc.) follows from the above arguments.

We point out that the resulting extended formulation is simple (a generalization of the union of polytopes [1]) and requires only $O(n)$ extra variables. Moreover, even though it might have exponentially many Edmonds' inequalities, they are separable in polytime [44]. Thus one can write an explicit linear formulation of the problem that could also be used as a strong relaxation for the variation of the stable set problem in claw-free graph with additional side constraints. One should also observe that if there would exist a compact extended description of the matching polytope, a well-known open problem, then also this formulation would be compact.

Faenza, Oriolo and Stauffer [18] gave another extended formulation of $STAB(G)$, G claw-free, that is better suited for projection, as it requires only one additional variable per strip. They derived from the projection of this formulation on the original space an alternative view to Theorem 14 and, more important, a polytime separation routine for $STAB(G)$ (in the original space).

Theorem 16 ([18]). *Let $G(V, E)$ be a claw-free graph. It is possible to separate in polynomial time over $STAB(G)$ using only a separation routine for matching and the solution of $O(|V|)$ compact linear programs.*

5 Open Questions

5.1 Complete Linear Description of $STAB(G)$ in the Original Space

It follows from [12] and Theorem 14, that in order to provide a linear description of the stable set polytope of *any* claw-free graph,

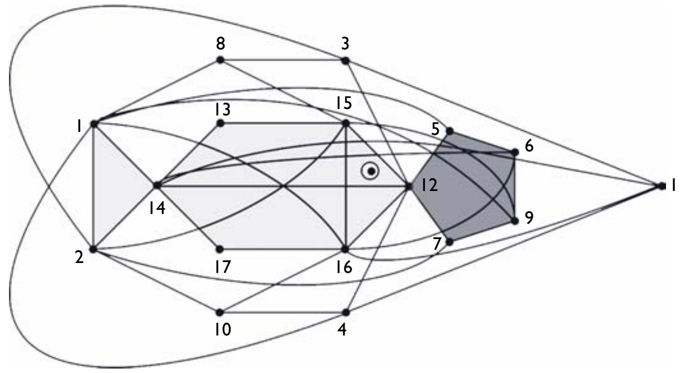

Figure 4. *The complement of a claw-free graph G. The graph G induces the facet: $2(x_{14} + x_{15} + x_{16}) + 3(x_1 + x_2 + x_3 + x_4) + 4(x_5 + x_6 + x_7 + x_9 + x_{12}) + 5(x_8 + x_{10} + x_{11}) + 6(x_{13} + x_{17}) \leq 8$. Note that G is not quasi-line and that $\alpha(G) = 3$. (The picture is a courtesy of Tristram Bogart, Annie Raymond and Rekha Thomas.)*

we are left with characterizing the stable set polytopes of claw-free graphs with stability number at most 3. In fact, following Theorem 12 we may restrict to claw-free, non-quasi-line graphs with stability number at most 3. However, even as Cook [14, 52, 33] characterized the stable set polytope of any graph G with $\alpha(G) \leq 2$, it seems that characterizing the stable set polytopes of graphs with stability number at most 3, is quite challenging, even if we restrict to claw-free non-quasi-line graphs. In fact, we already pointed out in Section 2.2 that for claw-free, but non quasi-line, graphs with stability number 3 there exist facets with arbitrarily many coefficients [46]; see Figure 4 for a facet inducing inequality with 5 different coefficients. This is not the case for quasi-line graphs (see Theorem 12) or claw-free graphs with stability number at least four and no clique-cutsets (see Theorem 13).

Pêcher and Wagler [45] worked on this question. While providing some better understanding of affine independence for the remaining difficult facets of SSP in claw-free graphs – the so-called *co-spanning forest structure* – this work leaves the full characterization in the original space still open. Indeed, Pêcher and Wagler [45] do not provide a 'proper' construction to produce a valid inequality associated with a given structure besides, basically, exploiting the polar of the polytope.

Observe that, in case one can solve the question above, the essence of the complete linear description for claw-free graphs will be significantly different from that of the stable set polytope of quasi-line graphs, for which inequalities are defined "algebraically". This suggests that, even if the case $\alpha(G) \leq 3$ was solved, additional insight might still be needed to get an elegant description of the stable set polytope of claw free graphs, if any. The next question proposes another standpoint on the problem that might lead to a simpler description of the polytope.

5.2 Calvillo's Theorem and the Intersection Property

Let $STAB_k(G) := conv\{x \in \{0,1\}^{|V|} : x_u + x_v \leq 1, \forall \{u, v\} \in E$ and $\sum_{v \in V} x_v = k\}$. Theorem 4 shows that $STAB_k(G) = STAB(G) \cap \{x \in \mathbb{R}^{|V|} | \sum_{v \in V} x_v = k\}$ and it might suggest that the stable set polytope of claw-free graphs has a nicer interpretation when intersected with the hyperplanes $\{x \in \mathbb{R}^{|V|} | \sum_{v \in V} x_v = k\}$ for all integer k. This is indeed the case for quasi-line graphs. In fact, building upon Theorem 4, Theorem 7, Theorem 11 and some arguments from [17], one can show that:

Lemma 5.1. *Let $G(V, E)$ be a quasi-line graph. For every integer k, $STAB_k(G)$ can be described by non-negativity inequalities, clique inequalities, Edmonds' inequalities and $\sum_{v \in V} x_v = k$.*

Can we hope for a similar result for claw-free graphs? Because of 5-wheel structure, one can easily show that, in contrast to quasi-line graphs, for claw-free graphs rank inequalities are not enough to describe $STAB_k(G)$; however a complete characterization of $STAB_k(G)$ might still be simple.

5.3 Minimal Linear Description for the Stable Set Polytope of Quasi-Line Graphs

For the matching polytope, Edmonds and Pulleyblank [16] gave a description of the *facets* of the polytope, giving necessary and sufficient conditions for an odd set of vertices to induce a facet. While the Ben Rebea theorem gives a linear description of $STAB(G)$ when G is quasi-line, it does not provide necessary and sufficient conditions for a family of cliques $\mathcal{K}$ and an integer $p \geq 2$ to induce a clique family inequality that is facet inducing. The question is open but there are a few results in this direction [26, 53, 54, 42].

5.4 The Chvatal–Gomory Rank of the Stable Set Polytope of Quasi-Line Graphs

Sbihi [50] reported that Edmonds conjectured in 1973 that the stable set polytope of any claw-free graph G had Chvátal-Gomory rank (CG-rank in the following) one from $QSTAB(G)$. This was proven false by Giles and Trotter [27], who provided a facet-defining inequality for $STAB(G)$ with CG-rank two. The result was strengthened by Chvátal [13] who showed that there exist graphs with $\alpha(G) = 2$, and therefore claw-free, with CG-rank unbounded. Interestingly this construction does not extend to quasi-line graphs, as building upon Theorem 12 and results in [14, 52, 33], one may show that the CG-rank of a quasi line graph G with $\alpha(G) = 2$ is one. However, facet-defining inequalities with CG-rank two exist also for quasi-line graphs, as shown by Oriolo [40]. This raises the following questions: is the CG-rank unbounded for quasi-line graphs or is it bounded? (Actually, despite some efforts, we could not produce for quasi-line graphs facet-defining inequalities with CG-rank bigger than two). We mention that Pêcher and Wagler [47] studied the CG-rank of general clique family inequalities and gave some upper bounds. Unfortunately they do not seem to be tight.

5.5 Improving the Complexity

The weighted matching problem in a graph $H(W,F)$ can be solved in $O(|W|(|W|log|W| + |F|)-$time [20]. It follows that we can find a MWSS in a line graph $G(V,E)$ in time $O(|V|^2 \log |V|)$. Following the algorithm by Faenza, Oriolo and Stauffer presented in Section 4.1 a MWSS in a claw-free graph $G(V,E)$ can be found in time $O(|V|(|V|\log|V|+|E|))$, i.e., slightly worse than for line graphs: can we close this gap? We believe that this should be doable, in particular for quasi-line graphs. Also note that the above algorithm uses only elementary data structures, so one could try to lower its complexity by using more sophisticated data structures.

5.6 A "Short" Proof of the Ben Rebea Theorem

One should note that Theorem 9 is quite close to Theorem 7. However, while the former theorem has a rather simple and direct proof, the proof of the latter relies on the general structure of claw-free graphs. We ask therefore whether it is possible to sharpen Theorem 9 so as to prove the same characterization of Theorem 7. The question of having a direct proof of Theorem 7 was raised already by King [31]. Note that such a proof, together with the proof of Theorem 11 and the proof that the Ben Rebea conjecture holds for fuzzy circular interval graphs in [17], would provide a "short" proof of Theorem 12.

Acknowledgements

We are grateful to Sam Burer, Yuri Faenza and Volker Kaibel for comments on a draft of this article and to Tristram Bogart, Annie Raymond and Rekha Thomas for allowing us to use Figure 1 and Figure 4.

Gianpaolo Oriolo, Dipartimento di Informatica, Sistemi e Produzione, Università Tor Vergata, Roma. oriolo@disp.uniroma2.it
Gautier Stauffer, Bordeaux Institute of Mathematics, 351 cours de la Liberation, 33405 Talence, France. gautier.stauffer@math.u-bordeaux1.fr
Paolo Ventura, Istituto di Analisi dei Sistemi ed Informatica "A. Ruberti" del CNR, Roma. paolo.ventura@iasi.cnr.it

References

[1] E. Balas. Disjunctive programming. In *Annals of Discrete Mathematics*, volume 5, pages 3–51, 1979.

[2] F. Barahona and A.R. Mahjoub. Compositions of graphs and polyhedra I: balanced induced subgraphs and acyclic subgraphs. *SIAM J. on Discr. Math.*, 7:344–358, 1994.

[3] F. Barahona and A.R. Mahjoub. Compositions of graphs and polyhedra II: stable sets. *SIAM J. on Discr. Math.*, 7:359–371, 1994.

[4] C. Berge. *Graphs and Hypergraphs*. Dunod, Paris, 1973.

[5] G. Calvillo. The concavity and intersection properties for integral polyhedra. In *Annals of Discrete Mathematics*, volume 8, 1981.

[6] M. Chudnovsky and P. Seymour. Quasi-line graphs. submitted.

[7] M. Chudnovsky and P. Seymour. Claw-free graphs, the stable set polyhedron, and Ben Rebea's conjecture. France – Israel Expert Workshop on Graph Classes and Graph Algorithms, 2004.

[8] M. Chudnovsky and P. Seymour. The structure of claw-free graphs. In *Surveys in Combinatorics*, volume 327 of *London Math. Soc. Lecture Notes*. 2005.

[9] M. Chudnovsky and P. Seymour. Claw-free graphs IV: Decomposition theorem. *J. Comb. Th. B*, 98:839–938, 2008.

[10] M. Chudnovsky and P. Seymour. Claw-free graphs V: Global structure. *J. Comb. Th. B*, 98:1373–1410, 2008.

[11] V. Chvátal. Edmonds polytopes and a hierarchy of combinatorial problems. In *Proceedings of the 4th Southeast Conf. Graph Theory Comput.*, pages 305–337, 1973.

[12] V. Chvátal. On certain polytopes associated with graphs. *J. Comb. Th. B*, 18:138–154, 1975.

[13] V. Chvátal. Cutting-plane proofs and the stability number of a graph. Technical Report 84326, University of Bonn, 1984.

[14] W. Cook. Personal communication, 2003.

[15] J. Edmonds. Maximum matching and a polyhedron with $0,1$ vertices. *J. Res. of Nat. Bureau of Stand. B*, 69B:125–130, 1965.

[16] J. Edmonds and W.R. Pulleyblank. Facets of 1-matching polyhedra. In C. Berge and D.R. Chuadhuri, editors, *Hypergraph Seminar*, pages 214–242, 1974.

[17] F. Eisenbrand, G. Oriolo, G. Stauffer, and P. Ventura. The stable set polytope of quasi-line graphs. *Combinatorica*, 28(1):45–67, 2008.

[18] Y. Faenza, G. Oriolo, and G. Stauffer. The hidden matching structure of the composition of strips: a polyhedral perspective. Technical Report RR-16 DII, Università di Roma Tor Vergata, 2010.

[19] Y. Faenza, G. Oriolo, and G. Stauffer. A $O(n^3)$ algorithm for the stable set problem in claw-free graphs. In Dana Randall, editor, *Proceedings of SODA 2011 Conference*, volume 630–646, 2011.

[20] H. N. Gabow. Data structures for weighted matching and nearest common ancestors with linking. In *Proceedings of the 1st Annual ACM-SIAM Symposium on Discrete Algorithms*, 1990.

[21] A. Galluccio, C. Gentile, and P. Ventura. Gear composition and the stable set polytope. *Operations Research Letters*, 36:419–423, 2008.

[22] A. Galluccio, C. Gentile, and P. Ventura. 2-clique-bond of stable set polyhedra. Technical Report 09-10, IASI-CNR, 2009. New version available on Optimization Online.

[23] A. Galluccio, C. Gentile, and P. Ventura. Gear composition of stable set polytopes and $\mathcal{G}$-perfection. *Mathematics of Operations Research*, 34:813–836, 2009.

[24] A. Galluccio, C. Gentile, and P. Ventura. The stable set polytope of claw-free graphs with large stability number I: Fuzzy antihat graphs are $\mathcal{W}$-perfect. Technical Report 10-06, IASI-CNR, 2010. Submitted.

[25] A. Galluccio, C. Gentile, and P. Ventura. The stable set polytope of claw-free graphs with large stability number II: Striped graphs are $\mathcal{G}$-perfect. Technical Report 10-07, IASI-CNR, 2010. Submitted.

[26] A. Galluccio and A. Sassano. The rank facets of the stable set polytope for claw-free graphs. *J. Comb. Th. B*, 69:1–38, 1997.

[27] R. Giles and L.E. Trotter Jr. On stable set polyhedra for $K_{1,3}$-free graphs. *J. Comb. Th. B*, 31:313–326, 1981.

[28] M. Grötschel, L. Lovász, and A. Schrijver. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 1(2):169–197, 1981.

[29] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric algorithms and combinatorial optimization*. Springer Verlag, Berlin, 1988.

[30] D. Hermelin, M. Mnich, E.J. Van Leeuwen, and G.J. Woeginger. Domination when the stars are out. In *Proceedings of ICALP 2011*, 2011.

[31] A. King. *Claw-free graphs and two conjectures on ω, Δ, and χ*. PhD thesis, McGill University, 2009.

[32] J. Krausz. Demonstration nouvelle d'une theoreme de whitney sur les reseaux. *Math. Fis. Lapok*, 50:75–89, 1943.

[33] T.M. Liebling, G. Oriolo, B. Spille, and G. Stauffer. On non-rank facets of the stable set polytope of claw-free graphs and circulant graphs. *Math. Methods of Oper. Research*, 59:25–35, 2004.

[34] L. Lovász and M.D. Plummer. Matching theory. *Annals of Discr. Math.*, 29, 1986.

[35] G.J. Minty. On maximal independent sets of vertices in claw-free graphs. *J. Comb. Th. B*, 28:284–304, 1980.

[36] D. Nakamura and A. Tamura. A revision of Minty's algorithm for finding a maximum weighted stable set of a claw-free graph. *Jour. OR Society of Japan*, 44:194–204, 2001.

[37] G.L. Nemhauser and L.E. Trotter Jr. Properties of vertex packing and independence system polyhedra. *Mathematical Programming, Series A*, 6:48–61, 1974.

[38] G.L. Nemhauser and L.E. Trotter Jr. Vertex packing: structural properties and algorithms. *Mathematical Programming, Series A*, 8:232–248, 1975.

[39] P. Nobili and A. Sassano. A reduction algorithm for the weighted stable set problem in claw-free graphs. In *Proceedings of the 10-th Cologne-Twente Workshop*, pages 223–226, 2011.

[40] G. Oriolo. Clique family inequalities for the stable set polytope for quasi-line graphs. *Discrete Applied Mathematics*, 132(3):185–201, 2003.

[41] G. Oriolo, U. Pietropaoli, and G. Stauffer. A new algorithm for the maximum weighted stable set problem in claw-free graphs. In *Proceedings of the 12th IPCO Conference*, volume 5035, pages 77–96, 2008.

[42] G. Oriolo and G. Stauffer. Clique-Circulant for the Stable Set Polytope of Quasi-line Graphs. *Mathematical Programming, Series A*, 2008.

[43] M. Padberg. On the facial structure of set packing polyhedra. *Mathematical Programming*, 5:199–215, 1973.

[44] M. W. Padberg and M. R. Rao. Odd minimum cut-sets and b-matchings. *Mathematics of Operations Research*, 7:67–80, 1982.

[45] A. Pêcher, U. Pietropaoli, and A. Wagler. Personal communication, 2006.

[46] A. Pêcher and A. K. Wagler. On facets of stable set polytopes of claw-free graphs with stability number 3. *Discrete Mathematics*, 310(3):493–498, 2010.

[47] A. Pêcher and A.K. Wagler. A note on the chv·tal-rank of clique family inequalities. *RAIRO – Operations Research*, 41:289–294, 2007.

[48] W.R. Pulleyblank and B. Shepherd. Formulations of the stable set polytope. In G. Rinaldi and L. Wolsey, editors, *Proceedings Third IPCO Conference*, pages 267–279, 1993.

[49] A. Ben Rebea. *Étude des stables dans les graphes quasi-adjoints*. PhD thesis, Université de Grenoble, 1981.

[50] N. Sbihi. Algorithme de recherche d'un stable de cardinalité maximum dans un graphe sans étoile. *Discrete Mathematics*, 29:53–76, 1980.

[51] A. Schrijver. *Combinatorial optimization*. Springer Verlag, Berlin, 2003.

[52] F.B. Shepherd. Applying lehman's theorems to packing problems. *Mathematical Programming*, 71(353–367), 1995.

[53] G. Stauffer. *On the stable set polytope of claw-free graphs*. PhD thesis, EPF Lausanne, 2005.

[54] G. Stauffer. On the facets of the stable set polytope of quasi-line graphs. *Operations Research Letters*, 39(3):208–212, 2011.

Discussion Column

Manfred W. Padberg

Node Packings in Graphs and Claw Free Graphs

When Volker Kaibel called me a couple of weeks ago to ask me to write a short historical note (as a discussion column to the article by Gianpaolo Oriolo, Gautier Stauffer, and Paolo Ventura) about node packings in graphs and especially in claw-free graphs, my reaction was more or less: Boy, it has been something like forty years ago that I've worked on that stuff! But I promised to do it and so here it is. Most of what I have to say is about node packings in graphs and the general context in which it happened, but some of it may help to explain how the interest in claw-free graphs travelled from Pittsburgh via Berlin, Bonn and New York to Rome.

1 Beginnings

Had you asked me back in 1969/1970 about node packing or vertex packing in graphs (bull-free, claw-free or whatever), you would have drawn a blank stare: I would not have known what you were talking about. These terms may have existed in graph theory, but not in integer programming. In integer programming we were concerned with linear programs with practical applications: airline crew scheduling problems, knapsack problems (with their roots in capital rationing in finance) and traveling salesman problems (TSPs with their roots in K. Menger's "Botenproblem" from the 1930's), to name just three favorite problems of mine in those days. The intimate connections between graph theoretical and integer or zero-one programming problems – that you young guys are all familiar with today – just had not been established yet. All right, to understand TSPs you need a modicum of graph theory, but that's all. What got my interest in this field were airline crew scheduling problems, perhaps because in September 1968 I had left my native Westphalia in Germany to fly to New York and then, by my ex-wife's car, to Pittsburgh, PA, where I had obtained a Ford Foundation Fellowship to complete my doctoral studies at Carnegie-Mellon University's GSIA and where I specialized with Egon Balas, see my historical note *"Mixed-integer Programming – 1968 and thereafter" in* Annals of Operations Research*, 2007, 149: 163–175*. Despite Dantzig, Fulkerson and S. Johnson's *milestone* 1954 paper *Solution of a Large Scale Travelling Salesman Problem* published in *Operations Research* – a world record as it solved a problem in 1,128 zero-one variables to optimality – computational integer programming had a bad name. Egon Balas' 1965 paper *An Additive Algorithm for Solving Linear Programming Programs with Zero-One Variables* published in *Operations Research* had bettered the picture somewhat, but all of this was "overshadowed" by Ralph Gomory's algebraic, some said "elegant", algorithm for integer programming (first abstracted in 1958 and published in 1963), which computationally just did not work. Of the many references in the literature to this effect, let me just mention Don Knuth's 1961 paper *Minimizing Drum Latency Time* in the *Journal of the Association for Computing*

Machinery and some articles in the book edited by Muth and Thompson *Industrial Scheduling, Prentice-Hall, 1963*. In any case, the recent "revival" of Gomory's mixed-integer cuts in computational integer programming does not contradict what I'm saying because they are based on "disjunctive" reasoning and thus different from the original Gomory cuts developped in 1958 or so; see also my paper *Classical Cuts for Mixed-Integer Programming and Branch-and-Cut* in *Math. Meth. Oper. Res. 2001, 53: 173–203* or its reprint in *Annals of O.R. 2005, 139: 321–352*. So much for the history as I found it back in 1968 in Pittsburgh, PA.

2 Step I

By April/May of 1971 I had finished and defended my PhD thesis *Essays in Integer Programming* at GSIA and prepared my return to Germany, because of my obligation to do so under the conditions of my Ford Foundation Fellowship. I wound up at the International Institute of Management in Berlin with a three-year contract. On March 25, 1970, Egon and I had submitted Chapter 2 of my thesis *On the Set-Covering Problem* to *Operations Research* where it appeared in the November-December 1972 issue. Chapter 3 of my thesis *'Simple' Zero-One Problems: Set Covering, Matchings and Coverings in Graphs* was widely distributed as the *Management Sciences Report No. 235* of CMU's GSIA and submitted to *Mathematical Programming* sometime in late 1971. In it I had abstracted from Egon's and mine results of Chapter 2 a property of the said problems called 'Simplicity' of a polytope (which I myself found in early 1972 to be erroneous).

In any case, my Chapter 3 contained the first results on the facets of these polytopes, namely the clique and 'lifted' odd-cycle facets. The pertaining correct results of it were published in somewhat improved form in *On the Facial Structure of Set Packing Polyhedra, Mathematical Programming, 1973, 5: 199–215*. Just for completeness, Chapter 4 of my thesis *Equivalent Knapsack-type Formulations of Bounded Integer Linear Programs: An Alternative Approach* appeared in *Naval Research Logistics Quarterly, 1972, 19: 699–708* and one of the technical appendices *A Remark on "An Inequality for the Number of Lattice Points in a Simplex"* in *SIAM Journal of Applied Math., 1971, 20: 638–641*. Another technical appendix of my thesis contained some results on "adjacent vertices cuts", but these were just again "cuts" and not "facet-defining cutting-planes" and thus I never published that stuff. *Voila*, that was essentially the content of my 1971 thesis. So much for those who still recently asked themselves what my thesis was all about. Just read the published stuff.

3 Step II

I had wanted to test facet-defining cutting-planes for node and set packing problems computationally in Berlin, but there were just no adequate computing facilities in Berlin and also a lack of test problems. Karla Hoffman and I did so eventually in our paper *Solving Airline Crew Scheduling Problems by Branch-and-Cut* published in *Management Science, 1993, 39: 657–682*. Needless to say, it worked wonderfully, but here I am jumping way ahead of time.

I landed a job at New York University's Graduate School of Business as of September 1974 when my obligation to the Ford Foundation to stay in Germany was over. On my way from Berlin to New York I "stopped" for about half a year at Bernhard Korte's then new Institute for Operations Research and Econometrics at Bonn University, where I met Martin Groetschel. I don't think that it is necessary to recall our joint work on the traveling salesman polytope here. Besides the theoretical work, Martin's 1976 thesis contained the solution to optimality of a 120-city TSP using only facet-defining cutting-planes – that's a linear program in 7,140 zero-one variables and thus another world record! *Wunderbar*, because that is exactly what I had in mind when I started out in 1970 to look for facet-defining cutting-planes, rather than arbitrary "cuts" with no proven mathematical properties other than "validity" for the problem in question. I won't recall either in detail my computational work on the TSP (with S. Hong, then with H. Crowder and later with Giovanni Rinaldi) as well as on other problems pursued with the same goal – to prove "empirically" the value of facet-defining cutting-planes in actual computation. But I will recall my work with M. Ram Rao *Odd Minimum Cut-Sets and b-Matchings, Mathematics of Operations Research, 1982, 7: 67-80*, which we presented at the 1979 Mathematical Programming Symposium in Montreal. For in the meantime, the late Leonid Khachian had proved that linear programming problems could be solved in polynomial time by the ellipsoid method and his result had traveled to the West just around 1979. After my presentation, Jack Edmonds and Laslo Lovasz conjectured that Ram and I had just given another poly-time algorithm for b-matchings in graphs. This turned out to be true and Ram and I were delighted when Martin Groetschel and Olaf Holland showed in *A cutting-plane algorithm for minimum perfect 2-matching, Computing, 1987, 39: 327–344,* that our pure cutting-plane algorithm (using the simplex method, of course, instead of the ellipsoid algorithm) outperformed Edmonds' graphical algorithm in practical computation. I should add that purely graphical problems occur only rarely in practice and are frequently complicated by additional constraints such as capacity and/or capital constraints which necessitates a cutting-plane approach.

The major consequence of Khachian's ellipsoid method was the equivalence of *optimization* and *separation* in terms of poly-time solvability, a result that was obtained in early 1980 independently by three different groups of researchers: Groetschel, Lovasz and Schrijver, Karp and Papadimitriou and Padberg and Rao. (You'll find a proof of this equivalence, e.g., in Chapter 9 of my book *Linear Optimization and Extensions, 1995, 2nd ed. 1999, Springer Verlag*). This equivalence generalized, of course, Edmonds and Lovasz's conjecture mentioned above. But it also reinforced the "hunt" for facet-defining cutting-planes that had followed my initial findings that *facets* of the convex hull of the integer solutions could indeed be found for (some) integer and mixed-integer programs, like node covering, node packing, set packing problems and then knapsack problems, TSPs, etc.

4 Step III

Also in or around 1979 I learned that the late George Minty had generalized Edmonds' matching algorithm and found a poly-time algorithm for vertex packings in claw-free graphs, see *Journal of Combinatorial Theory B, 1980, 284-304*, and independently of him Najiba Sbihi, see *Algorithme de Recherche d'un Stable de Cardinalite Maximum dans un Graphe sans Etoile, Discrete Mathematics, 1980, 29: 53–76*, as well. Once the equivalence of optimization and separation had been established, given the poly-time solvability of weighted vertex packing in $K_{1,3}$-free graphs, the separation problem for the associated convex hull of node packings in such graphs had to be solvable in poly-time as well. Being an eternal optimist, I put the problem of finding a complete minimal linear description for this problem on my list of things to do, but never came around to attacking this problem alone.

Sometime in 1981/1982, the late Mario Lucertini of Rome's *Universita Tor Vergata* invited me to do a two-week intensive course on combinatorial optimization in Rome – it must have been in August of 1982, because it was awfully hot and the class room had no air-conditionning. I met through Lucertini Giovanni Rinaldi and Antonio Sassano, who had just re-joined the Italian CNR after having worked for a while in their own company. After one of my courses, the four of us discussed over a cool beer in one of the shady squares of Rome a possible visit of Antonio and Giovanni with me at New York University. Antonio Sassano came in 1983/1984 to work with me at NYU, Giovanni came a little while later. I suggested to Antonio to work on the facial structure of the polytope of vertex packings in $K_{1,3}$-free graphs, but perhaps due to the relatively short time that Antonio stayed with me in New York, we did not get enough substantial results on the problem to write a joint paper on it. When he returned to Rome, he took the problem along with him and the desire to solve it; just look at Sassano's homepage at the *Universita La Sapienza*, where several papers on this topic (with various coauthors) are listed. I am sure that Antonio Sassano has a lot to do with the progress made in Italy on this problem like the new algorithm by Faenza, Oriolo, and Stauffer and the new polyhedral results by Galluccio, Gentile, and Ventura. Personnally, what I find very interesting is, of course, a pure cutting-plane algorithm for this problem, like the one that Ram Rao and I found for b-matchings in graphs and that (*simplex method or ellipsoid method*, I don't care) is computationally efficient and permits other complicating constraints to be added. I am happy to hear that the separation routine via the new extended formulation due to Faenza, Oriolo, and Stauffer provides this.

Manfred W. Padberg, Emeritus Professor, New York University, Stern School of Business, 17 rue Vendôme 13007 Marseille, France
manfred4@wanadoo.fr

Obituary

Oleg Burdakov, John Dennis, and Jorge Moré

Charles G. Broyden, 1933–2011

Charles George Broyden, Sweden, 2002 (Photo: Oleg Burdakov)

Charles George Broyden was born February 2nd, 1933, in England. He received his degree in Physics from Kings College London in 1955. He spent the first ten years of his career in industry. In 1967, he moved to the University of Essex where he became a professor and, later, dean of the School of Mathematics. In 1986, he decided to retire early to become a traveling scholar, but in 1990, he accepted an appointment as a professor of numerical analysis at the University of Bologna.

Charles received international recognition for his seminal 1965 paper in Mathematics of Computation, in which he proposed two methods for solving systems of equations. They later became known as Broyden's methods. Another of his most important achievements was the derivation of the Broyden-Fletcher-Goldfarb-Shanno (BFGS) updating formula, one of the key tools used in optimization. Moreover, Charles was among those who derived the symmetric rank-one updating formula, and his name is also attributed to the Broyden family of quasi-Newton methods.

At Bologna, Charles shifted the focus of his research to numerical linear algebra and, in particular, to conjugate gradient methods and to the taxonomy of these methods. Some of the main results of that period are summarized in his 2004 book with M.T. Vespucci *Krylov Solvers for Linear Algebraic Systems*.

In recognition of his fundamental contributions to the development of optimization and numerical mathematics, the journal *Optimization Methods and Software* (OMS) established the Charles Broyden prize. It is awarded yearly for the best paper published in OMS.

When Charles learned about the prize, he modestly noted that "I discovered my algorithms because I was in the right place at the right time". Being in the right place at the right time once could be good luck, but if this happens several times, this clearly indicates talent. Indeed, one can hardly find a book on numerical optimization where the discoveries of Charles Broyden are not mentioned.

Charles Broyden died on May 20, 2011. We will remember him as a highly dedicated, modest, and honest researcher, respected by his many friends and collaborators around the world. We express our sympathy to his wife Joan, his children and grandchildren.

Announcements

Call for Nominations for the 2012 Beale-Orchard-Hays Prize

Nominations are invited for the 2012 Beale-Orchard-Hays Prize for excellence in computational mathematical programming that will be awarded at the International Symposium on Mathematical Programming to be held in Berlin in August 2012.

The Prize is sponsored by the Mathematical Optimization Society, in memory of Martin Beale and William Orchard-Hays, pioneers in computational mathematical programming. Nominated works must have been published between Jan 1, 2009 and Dec 31, 2011, and demonstrate excellence in any aspect of computational mathematical programming. "Computational mathematical programming" includes the development of high-quality mathematical programming algorithms and software, the experimental evaluation of mathematical programming algorithms, and the development of new methods for the empirical testing of mathematical programming techniques. Full details of prize rules and eligibility requirements can be found at http://www.mathopt.org/?nav=boh.

The 2012 Prize Committee consists of Michael Ferris (Chair), Philip Gill, Tim Kelley, and Jon Lee.

Nominations can be submitted electronically or in writing, and should include detailed publication details of the nominated work. Electronic submissions should include an attachment with the final published version of the nominated work. If done in writing, submissions should include four copies of the nominated work. Supporting justification and any supplementary material are strongly encouraged but not mandatory. The Prize Committee reserves the right to request further supporting material and justification from the nominees.

Nominations should be submitted to:

Prof. Michael Ferris, Computer Sciences Department, University of Wisconsin, 1210 West Dayton Street, Madison, WI 53706, USA
Email: ferris@cs.wisc.edu

The deadline for receipt of nominations is January 15, 2012.

ISMP 2012 in Berlin

The *21st International Symposium on Mathematical Programming (ISMP 2012)* will take place in Berlin, Germany, August 19–24, 2012. ISMP is a scientific meeting held every three years on behalf of the Mathematical Optimization Society (MOS). It is the world congress of mathematical optimization where scientists as well as industrial users of mathematical optimization meet in order to present the most recent developments and results and to discuss new challenges from theory and practice.

Conference Topics

The conference topics address all theoretical, computational and practical aspects of mathematical optimization including:

- integer, linear, nonlinear, semidefinite, conic and constrained programming
- discrete and combinatorial optimization
- matroids, graphs, game theory, network optimization
- nonsmooth, convex, robust, stochastic, PDE-constrained and global optimization
- variational analysis, complementarity and variational inequalities
- sparse, derivative-free and simulation-based optimization
- implementations and software
- operations research
- logistics, traffic and transportation, telecommunications, energy systems, finance and economics

Conference Venue

The Symposium will take place at the main building of TU Berlin in the heart of the city close to the Tiergarten park.

The opening ceremony will take place on Sunday, August 19, 2012, at the Konzerthaus on the historic Gendarmenmarkt which is considered one of the most beautiful squares in Europe. The opening session will feature the presentation of awards by the Mathematical Optimization Society accompanied by symphonic music. This is followed by the welcome reception with a magnificent view on Gendarmenmarkt.

The conference dinner will take place at the Haus der Kulturen der Welt ("House of the Cultures of the World") located in the Tiergarten park with a beer garden on the shores of the Spree river and a view on the German Chancellery.

Registration and Important Dates

The conference registration will open before December 2011. The abstract submission deadline will be April 15, 2012, the early registration deadline June 15, 2012.

In accordance with the new MOS membership fees policy, ISMP 2012 will offer three early registration rates for regular attendees (not students, not retired, and not lifetime members of MOS):

- Euro 340 including MOS membership for 2013
- Euro 390 including MOS membership for 2013 and 2014
- Euro 415 including MOS membership for 2013–2015

The early registration rates for retirees (not lifetime members of MOS) are

- Euro 190 including MOS membership for 2013
- Euro 215 including MOS membership for 2013 and 2014
- Euro 230 including MOS membership for 2013–2015

The early registration rate for students is Euro 160. The early registration rate for lifetime members of MOS is Euro 280. The registration rates for late registration (after June 15, 2012) will be higher (details to be announced).

Webpage etc.

More details (including registration rates, hotel prices, sponsorship opportunities, exhibits etc.) can be found on the conference web pages at www.ismp2012.org.

Main building of TU Berlin in the heart of the city close to the Tiergarten park (Photo: TU Berlin/Dahl)

Haus der Kulturen der Welt ("House of the Cultures of the World") located in the Tiergarten park (Photo: Christoph Eyrich)

Call for Site Pre-Proposals ISMP 2015

The Symposium Advisory Committee of the Mathematical Optimization Society issues a call for pre-proposals to organize and host ISMP 2015, the triennial International Symposium on Mathematical Programming.

ISMP is the flagship event of our society, regularly gathering over a thousand scientists from around the world. The conference will be held in or around the month of August, 2015. Hosting ISMP provides a vital service to the mathematical optimization community and often has a lasting effect on the visibility of the hosting institution. It also presents a significant challenge. This call for pre-proposals is addressed at local groups willing to take up that challenge.

Preliminary bids will be examined by the Symposium Advisory Committee (SAC), which will then issue invitations for detailed bids. The final decision will be made and announced during ISMP 2012 in Berlin. Member of the SAC are
Jeff Linderoth, Chair linderoth@wisc.edu,
Shabbir Ahmed, sahmed@isye.gatech.edu,
Kurt Anstreicher, kurt-anstreicher@uiowa.edu,
Anders Forsgren, andersf@kth.se,
Martine Labbe, mlabbe@ulb.ac.be,
Rüdiger Schultz ruediger.schultz@uni-due.de, and
Shuzhong Zhang, zhangs@umn.edu.

Preliminary bids should be brief and contain information pertaining to the

(1) Location,
(2) Facilities,
(3) Logistics: Accommodation and transportation, and
(4) Likely local organizers.

Further information can be obtained from any member of the advisory committee. Please address your preliminary bids until September 15, 2011 to Jeff Linderoth linderoth@wisc.edu.

Conic Optimization – Special Issue for *Pacific Journal of OPTIMIZATION*

Original research articles are solicited for a forthcoming issue of *Pacific Journal of OPTIMIZATION* dedicated to Conic Optimization. Potential articles may focus on (or relate to) any subset of the theoretical, computational and practical aspects of Conic Optimization.

Over the last two decades, there has been a significant advance in the research of interior point methods and conic optimization. Today conic optimization has emerged as a major computational paradigm and has made significant impact on numerous problems previously considered intractable or difficult to approximate. This special issue aims to solicit papers that provide original research on conic optimization theory and algorithms, including available software. Also welcome are papers that deal with emerging, meaningful applications; or that give friendly overviews of certain theoretically advanced conic optimization techniques relevant to applications.

Examples of topics that will be addressed in this special issue include, but are not limited to:

– Theory and algorithms for large scale conic optimization
– Computational study of conic optimization algorithms
– Relaxation techniques based on conic optimization
– Applications of conic optimization in science and engineering (e.g., image processing, compressed sensing, digital communications and networks)

Instructions for authors are available on the web:
http://www.ybook.co.jp/online/forauthors.htm

Please e-mail the pdf-file of your full manuscript with a cover message in plain text to one of the guest editors by *September 1, 2011.*
Zhi-Quan Luo, University of Minnesota, luozq@umn.edu
Levent Tunçel, University of Waterloo, ltuncel@math.uwaterloo.ca
Naihua Xiu, Beijing Jiaotong University, nhxiu@center.njtu.edu.cn

IMPRINT
Editor: Katya Scheinberg, Department of Industrial and Systems Engineering, Lehigh University, Bethlehem, PA, USA. katyascheinberg@gmail.com ■ **Co-Editors:** Samuel Burer, Department of Management Sciences, The University of Iowa, Iowa City, IA 52242-1994, USA. samuel-burer@uiowa.edu ■ Volker Kaibel, Institut für Mathematische Optimierung, Otto-von-Guericke Universität Magdeburg, Universitätsplatz 2, 39108 Magdeburg, Germany. kaibel@ovgu.de ■ **Founding Editor:** Donald W. Hearn ■ Published by the Mathematical Optimization Society. ■ Design and typesetting by Christoph Eyrich, Berlin, Germany. optima@0x45.de ■ Printed by Oktoberdruck AG, Berlin, Germany.

Application for Membership

I wish to enroll as a member of the Society. My subscription is for my personal use and not for the benefit of any library or institution.

☐ I will pay my membership dues on receipt of your invoice.
☐ I wish to pay by credit card (Master/Euro or Visa).

Credit card no. *Expiration date*

Family name

Mailing address

Telephone no. *Telefax no.*

E-mail

Signature

Mail to:
Mathematical Optimization Society
3600 Market St, 6th Floor
Philadelphia, PA 19104-2688
USA

Cheques or money orders should be made payable to The Mathematical Optimization Society, Inc. Dues for 2011, including subscription to the journal *Mathematical Programming*, are US $ 90. Retired are $ 45. Student applications: Dues are $ 22.50. Have a faculty member verify your student status and send application with dues to above address.

Faculty verifying status

Institution